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A VARIATIONAL THEORY OF POROUS MEDIA

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Abstract—A theory of porous elastic solids, either dry or saturated by a compressible fluid, is presented which is based upon Hamilton's extended variational principle. The theory uses a more extensive kinematical description of the materials than is used in the Biot theory and includes the effects of the local expansions and contractions of the porous structure (microinertial effects). Unlike Biot's formulation, the constitutive relations for the porous solid and for the fluid depend only upon kinematical variables associated with the solid and with the fluid, respectively. The relation of the present theory to the Biot theory is discussed and it is shown that the theories are equivalent in the limit of large wavelengths. Wave propagation results are presented based on data for Berea sandstone and it is shown that the microinertial effects substantially alter the dispersion and attenuation behavior predicted by the Biot theory.

1. INTRODUCTION

The authors have recently presented a theory for mixtures of immiscible fluids which is based upon Hamilton's extended variational principle[1]. The volume fractions of the constituents play a major role in the theory. The fact that the sum of the volume fractions of the constituents must equal one is introduced into the variational principle as a constraint and is thereby incorporated directly into the equations of motion. Furthermore, the kinetic energies associated with local changes in the volumes of the constituents (i.e. first-order microinertial effects) were included in the theory in analogy to continuum models for composite materials[2].

In the present paper the theory is extended to the case of a porous elastic material, either dry or saturated by a compressible fluid. The theory differs from the work of Biot[3–8], and from continuum theories of fluid-solid mixtures such as those presented in[9–13], in several respects:

(i) The kinematical information used is more extensive. In Biot's theory, the deformational states of the solid and of the fluid are expressed by the strain of the solid $e_{(s)jk}$ and the dilatation of the displacement vector of the fluid $e_{(f)} = u_{(f)k,k}$. In the continuum theories of fluid-solid mixtures cited above, the deformational states are expressed by the strain of the solid $e_{(s)jk}$ and the partial density of the fluid $\rho_{(f)}$ (the density of fluid per unit volume of the fluid-solid mixture). In the present work, the deformational states are expressed by the strain of the solid $e_{(s)jk}$, the volume fractions of the solid and of the fluid $\phi_{(s)}$, $\phi_{(f)}$, and the densities of the solid and of the fluid $\bar{\rho}_{(s)}$, $\bar{\rho}_{(f)}$. (The terms $\bar{\rho}_{(s)}$ and $\bar{\rho}_{(f)}$ are the actual material densities. They are related to the partial densities by $\rho_{(s)} = \phi_{(s)}\bar{\rho}_{(s)}$, $\rho_{(f)} = \phi_{(f)}\bar{\rho}_{(f)}$.)

(ii) The kinematical description which is used permits the volume fraction constraint to be introduced explicitly.

(iii) The kinematical description which is used also permits the constitutive relations for the porous solid and for the fluid to depend only upon kinematical variables associated with the solid and with the fluid, respectively.

(iv) The theory includes the kinetic energies of local expansion and contraction of the solid and of the fluid. A porous medium is intrinsically microstructural—the pores are the microstructure—and can undergo local expansion of the pores which is independent of the gross motion of the material. The present theory models this local motion by including in the variational principle kinetic energies that are in terms of the squares of the rates of change of the material densities $\bar{\rho}_{(s)}$ and $\bar{\rho}_{(f)}$[1].

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Theories of porous materials which include the volume fractions of the constituents have been presented by Garg[14], Garg *et al.*[15], and by Kenyon[16]. In these works, constitutive relations for the volume fractions are introduced. In the present theory, the volume fractions are independent kinematical variables which are obtained explicitly from the solutions of the equations. Furthermore, these works do not use a variational formulation and do not include microinertial effects.

It is shown that in the limit of large wavelengths, the equations of the present theory are equivalent to the Biot equations. However, by comparing solutions for the propagation of steady state compressible waves in Berea sandstone, it is shown that microinertial effects substantially change the dispersion and attenuation characteristics of the waves.

2. KINEMATICS

The notation and essential results required for the application of the variational methods of mechanics to a mixture have been presented previously[1] and will be summarized here.

The solid and fluid are treated as individual continua. Each continuum has a motion (in Cartesian tensor notation)

$$x_i = \chi_{(\xi)i}(\mathbf{X}_{(\xi)}, t), \tag{1}$$

where x_i is the position vector of the material particle of the ξ constituent ($\xi = s$ for the solid and $\xi = f$ for the fluid) whose position is $\mathbf{X}_{(\xi)}$ in a specified reference configuration. The velocity $v_{(\xi)i}$ and acceleration $a_{(\xi)i}$ of the ξ constituent are

$$v_{(\xi)i} = \frac{\partial}{\partial t}\chi_{(\xi)i}(\mathbf{X}_{(\xi)}, t), \tag{2}$$

$$a_{(\xi)i} = \frac{\partial^2}{\partial t^2}\chi_{(\xi)i}(\mathbf{X}_{(\xi)}, t). \tag{3}$$

The rate of change of a variable $\Gamma_{(\xi)}(\mathbf{x}, t)$ following the motion of the γ constituent is

$$\dot{\Gamma}_{(\xi)}|_{\mathbf{X}_{(\gamma)}} = \frac{\partial}{\partial t}\Gamma_{(\xi)} + \Gamma_{(\xi),i}v_{(\gamma)i}, \tag{4}$$

where, $i = (\partial/\partial x_i)$, and let

$$\dot{\Gamma}_{(\xi)} = \dot{\Gamma}_{(\xi)}|_{\mathbf{X}_{(\xi)}}. \tag{5}$$

The Jacobian of the motion (1) is

$$J_{(\xi)} = \det\left(\frac{\partial\chi_{(\xi)i}}{\partial X_{(\xi)k}}\right), \tag{6}$$

and the densities, partial densities, and volume fractions are related by

$$\rho_{(\xi)} = \phi_{(\xi)}\bar{\rho}_{(\xi)}, \tag{7}$$

where the volume fractions satisfy

$$\sum_{\xi}\phi_{(\xi)}(\mathbf{x}, t) = 1. \tag{8}$$

The conservation of mass equation for each constituent is

$$\dot{\rho}_{(\xi)} + \rho_{(\xi)}v_{(\xi)k,k} = 0, \tag{9}$$

or alternatively

$$J_{(\xi)} = \frac{\rho_{(\xi)0}}{\rho_{(\xi)}} = \frac{\phi_{(\xi)0}\bar{\rho}_{(\xi)0}}{\phi_{(\xi)}\bar{\rho}_{(\xi)}}, \tag{10}$$

where a subscript 0 denotes that a variable is evaluated in the reference configuration.

When a variation $\delta x_{(\xi)i}$ is added to the motion (1),

$$x_i = \chi_{(\xi)i}(\mathbf{X}_{(\xi)}, t) + \delta x_{(\xi)i}(\mathbf{X}_{(\xi)}, t), \tag{11}$$

the resulting variations in the velocity $v_{(\xi)i}$ and Jacobian $J_{(\xi)}$ are

$$\delta v_{(\xi)i} = \delta \dot{x}_{(\xi)i}, \tag{12}$$

$$\delta J_{(\xi)} = J_{(\xi)} \delta x_{(\xi)k,k}. \tag{13}$$

The variation of a function $\Gamma_{(\xi)}$ holding the spatial point fixed can be written in terms of the variation $\delta\Gamma_{(\xi)}$ holding the material point $\mathbf{X}_{(\xi)}$ fixed as

$$\delta\Gamma_{(\xi)}\big|_x = \delta\Gamma_{(\xi)} - \Gamma_{(\xi),i}\delta x_{(\xi)i}. \tag{14}$$

Hamilton's extended principle is written

$$\delta \int_{t_1}^{t_2} (T - U)\,\mathrm{d}t + \int_{t_1}^{t_2} \delta w\,\mathrm{d}t = 0, \tag{15}$$

for variations which vanish at the arbitrary times t_1, t_2, where T is the kinetic energy, U is the potential energy, and δw is the virtual work due to forces not represented by a potential.

3. FLUID SATURATED POROUS MEDIUM

Consider motions of a material volume v of the porous solid-fluid mixture such that $\delta x_{(s)i}$ and the normal component of $\delta x_{(f)i}$ vanish at the surface s of v.

The kinetic energy of the mixture contained in v is written

$$T = \sum_{\xi} \int_v \frac{1}{2}\rho_{(\xi)} v_{(\xi)k} v_{(\xi)k}\,\mathrm{d}v + \int_v \frac{1}{2} m(v_{(f)k} - v_{(s)k})(v_{(f)k} - v_{(s)k})\,\mathrm{d}v + \sum_{\xi} \int_v \frac{1}{2}\rho_{(\xi)} I_{(\xi)} \dot{\bar{\rho}}^2{}_{(\xi)}\,\mathrm{d}v. \tag{16}$$

The first two terms in (16) are the kinetic energy terms used by Biot[3]. In his notation, $m = -\rho_{12}$. The third term is the kinetic energy of local expansion of the constituents[1]. The estimation of the coefficients $I_{(\xi)}$ by considering the local expansional kinetic energy of a cylindrical pore is discussed in Appendix 1.

A specific strain energy $E_{(s)}$ is introduced for the porous solid which is assumed to be a function of the strain

$$e_{(s)kj} = \frac{1}{2}(u_{(s)k,j} + u_{(s)j,k}), \tag{17}$$

where $u_{(s)k}$ is the infinitesimal displacement vector†, and the density $\bar{\rho}_{(s)}$,

$$E_{(s)} = E_{(s)}(e_{(s)kj}, \bar{\rho}_{(s)}). \tag{18}$$

It is important to note that $\bar{\rho}_{(s)}$ can vary independently of the strain in a porous solid; this is an additional degree of freedom. For the fluid, the specific strain energy depends only on the

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†Although the variational procedure is well suited to deriving a nonlinear theory, the present paper is restricted to the linear theory.

density $\bar{\rho}_{(f)}$,

$$E_{(f)} = E_{(f)}(\bar{\rho}_{(f)}), \tag{19}$$

so that

$$U = \sum_{\xi} \int_{v} \rho_{(\xi)} E_{(\xi)}\,\mathrm{d}v. \tag{20}$$

The virtual work is written

$$\delta w = \sum_{\xi} \int_{v} (\rho_{(\xi)} f_{(\xi)k} + p_{(\xi)k})\delta x_{(\xi)k}\,\mathrm{d}v, \tag{21}$$

where $f_{(\xi)k}$ is an external body force density (such as gravity), and $p_{(\xi)k}$ is the interaction (drag) force on the ξ constituent.

Including eqns (8) and (10) as constraints and using (16), (20) and (21), Hamilton's principle (15) is, introducing Lagrange multipliers λ and $\mu_{(\xi)}$,

$$\begin{aligned}\int_{t_1}^{t_2} \Bigg\{ \delta \Bigg[\sum_{\xi} \int_{v} \frac{1}{2} \rho_{(\xi)} v_{(\xi)k} v_{(\xi)k}\,\mathrm{d}v &+ \int_{v} \frac{1}{2} m (v_{(f)k} - v_{(s)k})(v_{(f)k} - v_{(s)k})\,\mathrm{d}v \\ &+ \sum_{\xi} \int_{v} \frac{1}{2} \rho_{(\xi)} I_{(\xi)} \dot{\bar{\rho}}^{2}_{(\xi)}\,\mathrm{d}v - \sum_{\xi} \int_{v} \rho_{(\xi)} E_{(\xi)}\,\mathrm{d}v \Bigg] \\ &+ \sum_{\xi} \int_{v} (\rho_{(\xi)} f_{(\xi)k} + p_{(\xi)k})\delta x_{(\xi)k}\,\mathrm{d}v - \int_{v} \lambda \delta \Big(\sum_{\xi} \phi_{(\xi)} - 1 \Big)\Big|_{k}\,\mathrm{d}v \\ &+ \sum_{\xi} \int_{v} \mu_{(\xi)} \delta \left(J_{(\xi)} - \frac{\phi_{(\xi)0} \bar{\rho}_{(\xi)0}}{\phi_{(\xi)} \bar{\rho}_{(\xi)}} \right) \mathrm{d}v \Bigg\}\,\mathrm{d}t = 0. \end{aligned} \tag{22}$$

Using (12)–(14) and integrating by parts, (22) can be written†

$$\begin{aligned}\int_{t_1}^{t_2} \Bigg\{ \sum_{\xi} \int_{v} -\rho_{(\xi)} a_{(\xi)k} \delta x_{(\xi)k}\,\mathrm{d}v &+ \int_{v} - m (a_{(f)k} - a_{(s)k})(\delta x_{(f)k} - \delta x_{(s)k})\,\mathrm{d}v \\ &+ \sum_{\xi} \int_{v} - \rho_{(\xi)} I_{(\xi)} \ddot{\bar{\rho}}_{(\xi)} \delta \bar{\rho}_{(\xi)}\,\mathrm{d}v - \int_{v} \left[- \left(\rho_{(s)} \frac{\partial E_{(s)}}{\partial e_{(s)kj}} \right)_{,j} \delta x_{(s)k} + \rho_{(s)} \frac{\partial E_{(s)}}{\partial \bar{\rho}_{(s)}} \delta \bar{\rho}_{(s)} \right] \mathrm{d}v \\ &- \int_{v} \rho_{(f)} \frac{\mathrm{d}E_{(f)}}{\mathrm{d}\bar{\rho}_{(f)}} \delta \bar{\rho}_{(f)}\,\mathrm{d}v + \sum_{\xi} \int_{v} (\rho_{(\xi)} f_{(\xi)k} + p_{(\xi)k}) \delta x_{(\xi)k}\,\mathrm{d}v \\ &- \int_{v} \lambda \sum_{\xi} (\delta \phi_{(\xi)} - \phi_{(\xi),k} \delta x_{(\xi)k})\,\mathrm{d}v \\ &+ \sum_{\xi} \int_{v} \left[-(\mu_{(\xi)} J_{(\xi)})_{,k} \delta x_{(\xi)k} + \mu_{(\xi)} J_{(\xi)} \left(\frac{\delta \phi_{(\xi)}}{\phi_{(\xi)}} + \frac{\delta \bar{\rho}_{(\xi)}}{\bar{\rho}_{(\xi)}} \right) \right] \mathrm{d}v \Bigg\}\,\mathrm{d}t = 0. \end{aligned} \tag{23}$$

Requiring (23) to be satisfied under independent variations $\delta x_{(\xi)k}$, $\delta \bar{\rho}_{(\xi)}$, $\delta \phi_{(\xi)}$ gives the equations

$$\rho_{(s)} a_{(s)k} = m(a_{(f)k} - a_{(s)k}) + \left(\rho_{(s)} \frac{\partial E_{(s)}}{\partial e_{(s)kj}} \right)_{,j} + \rho_{(s)} f_{(s)k} + p_{(s)k} + \lambda \phi_{(s),k} - (\mu_{(s)} J_{(s)})_{,k}, \tag{24}$$

$$\rho_{(f)} a_{(f)k} = - m(a_{(f)k} - a_{(s)k}) + \rho_{(f)} f_{(f)k} + p_{(f)k} + \lambda \phi_{(f),k} - (\mu_{(f)} J_{(f)})_{,k}, \tag{25}$$

$$\rho_{(s)} I_{(s)} \ddot{\bar{\rho}}_{(s)} = - \rho_{(s)} \frac{\partial E_{(s)}}{\partial \bar{\rho}_{(s)}} + \frac{\mu_{(s)} J_{(s)}}{\bar{\rho}_{(s)}}, \tag{26}$$

$$\rho_{(f)} I_{(f)} \ddot{\bar{\rho}}_{(f)} = - \rho_{(f)} \frac{\partial E_{(f)}}{\partial \bar{\rho}_{(f)}} + \frac{\mu_{(f)} J_{(f)}}{\bar{\rho}_{(f)}}, \tag{27}$$

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†The determination of the variation of the relative velocity kinetic energy term is discussed in Appendix 2.

$$\lambda = \frac{\mu_{(s)}J_{(s)}}{\phi_{(s)}}, \tag{28}$$

$$\lambda = \frac{\mu_{(f)}J_{(f)}}{\phi_{(f)}}. \tag{29}$$

Writing the density as the sum of the reference value and a small perturbation,

$$\bar{\rho}_{(\xi)} = \bar{\rho}_{(\xi)0} + \tilde{\bar{\rho}}_{(\xi)}, \tag{30}$$

the strain energies (18) and (19) can be written as isotropic second-order expansions in the forms

$$\rho_{(s)}E_{(s)} = \frac{1}{2}\, c e_{(s)} e_{(s)} + d e_{(s)jk} e_{(s)jk} + f e_{(s)} \tilde{\bar{\rho}}_{(s)} + \frac{1}{2}\, g \tilde{\bar{\rho}}_{(s)} \tilde{\bar{\rho}}_{(s)}, \tag{31}$$

$$\rho_{(f)}E_{(f)} = \frac{1}{2}\, h \tilde{\bar{\rho}}_{(f)} \tilde{\bar{\rho}}_{(f)}, \tag{32}$$

where $e_{(s)} = e_{(s)kk}$ and c, d, f, g and h are constitutive constants. Note that c, d, f and g depend only on the properties of the porous solid while h depends only on the fluid.

Substituting (31) and (32) into (24)–(27), assuming the interaction force to be of the form

$$p_{(s)k} = -p_{(f)k} = b(\dot{u}_{(f)k} - \dot{u}_{(s)k}), \tag{33}$$

where b is a constant, and using (28) and (29) to eliminate $\mu_{(\xi)}J_{(\xi)}$, the linearized equations are

$$\rho_{(s)0}\ddot{u}_{(s)k} = m(\ddot{u}_{(f)k} - \ddot{u}_{(s)k}) + c e_{(s),k} + 2d e_{(s)jk,j} + f \tilde{\bar{\rho}}_{(s),k} + b(\dot{u}_{(f)k} - \dot{u}_{(s)k}) - \phi_{(s)0}\lambda_{,k}, \tag{34}$$

$$\rho_{(f)0}\ddot{u}_{(f)k} = -m(\ddot{u}_{(f)k} - \ddot{u}_{(s)k}) - b(\dot{u}_{(f)k} - \dot{u}_{(s)k}) - \phi_{(f)0}\lambda_{,k}, \tag{35}$$

$$\bar{\rho}^2_{(s)0} I_{(s)0} \ddot{\tilde{\bar{\rho}}}_{(s)} = -\frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}} f e_{(s)} - \frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}} g \tilde{\bar{\rho}}_{(s)} + \lambda, \tag{36}$$

$$\bar{\rho}^2_{(f)0} I_{(f)0} \ddot{\tilde{\bar{\rho}}}_{(f)} = -\frac{\bar{\rho}_{(f)0}}{\phi_{(f)0}} h \tilde{\bar{\rho}}_{(f)} + \lambda, \tag{37}$$

where the external body force density has been deleted.

Writing the volume fraction as

$$\phi_{(\xi)} = \phi_{(\xi)0} + \tilde{\phi}_{(\xi)}, \tag{38}$$

the linearized conservation of mass equations (10) and volume fraction constraint (8) are

$$-e_{(s)} = \frac{\tilde{\phi}_{(s)}}{\phi_{(s)0}} + \frac{\tilde{\bar{\rho}}_{(s)}}{\bar{\rho}_{(s)0}}, \tag{39}$$

$$-e_{(f)} = \frac{\tilde{\phi}_{(f)}}{\phi_{(f)0}} + \frac{\tilde{\bar{\rho}}_{(f)}}{\bar{\rho}_{(f)0}}, \tag{40}$$

$$\tilde{\phi}_{(s)} + \tilde{\phi}_{(f)} = 0. \tag{41}$$

Eliminating $\tilde{\phi}_{(s)}$, $\tilde{\phi}_{(f)}$ from (39) to (41) yields the equation

$$-\phi_{(s)0}e_{(s)} - \phi_{(f)0}e_{(f)} = \frac{\phi_{(s)0}}{\bar{\rho}_{(s)0}}\tilde{\bar{\rho}}_{(s)} + \frac{\phi_{(f)0}}{\bar{\rho}_{(f)0}}\tilde{\bar{\rho}}_{(f)}. \tag{42}$$

This completes the development of the linear theory for the fluid saturated porous medium. Equations (34)–(37) and (42) provide a system of equations in the unknowns $u_{(s)k}$, $u_{(f)k}$, $\tilde{\bar{\rho}}_{(s)}$, $\tilde{\bar{\rho}}_{(f)}$ and λ.

4. POROUS MEDIUM

In the absence of the fluid, the Hamilton's principle (22) reduces to

$$\int_{t_1}^{t_2}\left\{\delta\left[\int_v \frac{1}{2}\rho_{(s)}v_{(s)k}v_{(s)k}\,\mathrm{d}v+\int_v \frac{1}{2}\rho_{(s)}I_{(s)}\dot{\bar{\rho}}_{(s)}^2\,\mathrm{d}v-\int_v \rho_{(s)}E_{(s)}\,\mathrm{d}v\right]+\int_v \rho_{(s)}f_{(s)k}\delta x_{(s)k}\,\mathrm{d}v\right\}\mathrm{d}t=0, \quad (43)$$

which leads to the equations

$$\rho_{(s)}a_{(s)k}=\left(\rho_{(s)}\frac{\partial E_{(s)}}{\partial e_{(s)kj}}\right)_{,j}+\rho_{(s)}f_{(s)k}, \quad (44)$$

$$\rho_{(s)}I_{(s)}\ddot{\bar{\rho}}_{(s)}=-\rho_{(s)}\frac{\partial E_{(s)}}{\partial\bar{\rho}_{(s)}}. \quad (45)$$

Using the expansion (31), the linearized equations in the absense of the external body force are

$$\rho_{(s)0}\ddot{u}_{(s)k}=ce_{(s),k}+2de_{(s)jk,j}+f\tilde{\bar{\rho}}_{(s),k}, \quad (46)$$

$$\bar{\rho}_{(s)0}^2I_{(s)0}\ddot{\tilde{\bar{\rho}}}_{(s)}=-\frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}}fe_{(s)}-\frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}}g\tilde{\bar{\rho}}_{(s)}, \quad (47)$$

and the linearized conservation of mass equation is

$$-e_{(s)}=\frac{\tilde{\phi}_{(s)}}{\phi_{(s)0}}+\frac{\tilde{\bar{\rho}}_{(s)}}{\bar{\rho}_{(s)0}}. \quad (48)$$

Equations (46) and (47) provide equations in the variables $u_{(s)k}$, $\tilde{\bar{\rho}}_{(s)}$. Equation (48) then determines $\tilde{\phi}_{(s)}$. Note that the linear momentum equation (46) and the expansional equation (47) are coupled. Also, it should be emphasized that the coefficients c, d, f, g in (46) and (47) will have the same values for a particular porous material as the corresponding coefficients which appear in (34)–(36).

5. RELATION TO THE BIOT THEORY

Let it be assumed that the expansional inertia coefficients $I_{(\xi)0}=0$ or that the wavelengths being considered are very large compared to the pore structure so that the effects of the $I_{(\xi)0}$ are negligible. Then (36), (37) and (42) are algebraic equations which can be solved for $\tilde{\bar{\rho}}_{(s)}$, $\tilde{\bar{\rho}}_{(f)}$ and λ in terms of the dilatational strains $e_{(s)}$ and $e_{(f)}$. The results are

$$\tilde{\bar{\rho}}_{(s)}=\frac{1}{D}(C_{11}e_{(s)}+C_{12}e_{(f)}), \quad (49)$$

$$\tilde{\bar{\rho}}_{(f)}=\frac{1}{D}(C_{21}e_{(s)}+C_{22}e_{(f)}), \quad (50)$$

$$\lambda=\frac{\bar{\rho}_{(f)0}h}{\phi_{(f)0}D}(C_{21}e_{(s)}+C_{22}e_{(f)}), \quad (51)$$

where

$$D=h+q^2r^2g, \quad (52)$$

$$C_{11}=-\bar{\rho}_{(s)0}h-q^2r^2f, \quad (53)$$

$$C_{12}=-\bar{\rho}_{(s)0}rh, \quad (54)$$

$$C_{21}=-\bar{\rho}_{(s)0}qrg+qrf, \quad (55)$$

$$C_{22}=-\bar{\rho}_{(s)0}qr^2g, \quad (56)$$

$$q=\bar{\rho}_{(s)0}/\bar{\rho}_{(f)0}, \quad (57)$$

$$r=\phi_{(f)0}/\phi_{(s)0}. \quad (58)$$

Substituting (49)–(51) into eqns (34) and (35) gives the two equations

$$\rho_{(s)0}\ddot{u}_{(s)k} = m(\ddot{u}_{(f)k} - \ddot{u}_{(s)k}) + Ae_{(s),k} + 2Ne_{(s)jk,j} + Qe_{(f),k} + b(\dot{u}_{(f)k} - \dot{u}_{(s)k}), \tag{59}$$

$$\rho_{(f)0}\ddot{u}_{(f)k} = -m(\ddot{u}_{(f)k} - \ddot{u}_{(s)k}) + Qe_{(s),k} + Re_{(f),k} - b(\dot{u}_{(f)k} - \dot{u}_{(s)k}), \tag{60}$$

where

$$A = c + \frac{1}{D}\left(fC_{11} - \frac{\bar{\rho}_{(f)0}h}{r}C_{21}\right), \tag{61}$$

$$N = d, \tag{62}$$

$$Q = \frac{1}{D}\left(fC_{12} - \frac{\bar{\rho}_{(f)0}h}{r}C_{22}\right), \tag{63}$$

$$R = -\frac{\bar{\rho}_{(f)0}h}{D}C_{22}. \tag{64}$$

Equations (59) and (60) are the Biot equations[4]. Thus, although the present theory contains more degrees of freedom than are present in Biot's theory, the variational formulation verifies that the Biot equations are valid in the limit of large wavelengths. This procedure also provides an alternate derivation of Biot's equations.

Furthermore, this comparison provides a means for solving for the coefficients of the present theory in terms of the Biot coefficients. From (32), it is easy to show that h is related to the bulk modulus of the fluid $\lambda_{(f)}$ by

$$h = \phi_{(f)}\lambda_{(f)}/\bar{\rho}^2_{(f)}. \tag{64}$$

Then (61)–(64) can be solved for the coefficients c, d, f and g in terms of the Biot coefficients A, N, Q and R:

$$c = A + \frac{R}{r^2} - 2\frac{Q}{r} + \frac{(R - rQ)^2}{r^2(\lambda_{(f)}\phi_{(f)} - R)}, \tag{65}$$

$$d = N, \tag{66}$$

$$f = \frac{\lambda_{(f)}(R - rQ)}{r\bar{\rho}_{(s)0}\left(r\lambda_{(f)} - \frac{R}{\phi_{(s)0}}\right)}, \tag{67}$$

$$g = \frac{\lambda_{(f)}R}{r\bar{\rho}^2_{(s)0}\left(r\lambda_{(f)} - \frac{R}{\phi_{(s)0}}\right)}. \tag{68}$$

Of course, from (59) and (60) it is clear that the coefficients $m = -\rho_{12}$ and b in the present theory are the same as the corresponding coefficients in the Biot equations. Thus, in obtaining computational results from the present theory it is possible to use the available experimental data on the Biot coefficients[17–21].

A dividend of the present analysis is that, at wavelengths at which the effects of the $I_{(\xi)0}$ can be neglected, (49) and (50) permit the material densities to be calculated from solutions of Biot's equations, and the volume fractions can then be obtained from (39) and (40).

6. WAVE PROPAGATION RESULTS AND DISCUSSION

One dimensional steady state wave propagation results have been obtained by assuming solutions of the form

$$\Gamma_{(\xi)} = \hat{\Gamma}_{(\xi)}e^{i(\omega t - kx_1)}, \tag{69}$$

where $\hat{\Gamma}_{(\xi)}$ is a constant, ω is the frequency, and k is the wave number. The one dimensional forms of eqns (34)–(37) and (42) for the fluid saturated porous solid become

$$-\omega^2\rho_{(s)0}\hat{u}_{(s)} = -\omega^2 m(\hat{u}_{(f)}-\hat{u}_{(s)}) - k^2(c+2d)\hat{u}_{(s)} - ikf\hat{\bar{\rho}}_{(s)} + i\omega b(\hat{u}_{(f)}-\hat{u}_{(s)}) + ik\phi_{(s)0}\hat{\lambda}, \tag{70}$$

$$-\omega^2\rho_{(f)0}\hat{u}_{(f)} = \omega^2 m(\hat{u}_{(f)}-\hat{u}_{(s)}) - i\omega b(\hat{u}_{(f)}-\hat{u}_{(s)}) + ik\phi_{(f)0}\hat{\lambda}, \tag{71}$$

$$-\omega^2\bar{\rho}^2_{(s)0}I_{(s)0}\hat{\bar{\rho}}_{(s)} = ik\frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}}f\hat{u}_{(s)} - \frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}}g\hat{\bar{\rho}}_{(s)} + \hat{\lambda}, \tag{72}$$

$$-\omega^2\bar{\rho}^2_{(f)0}I_{(f)0}\hat{\bar{\rho}}_{(f)} = -\frac{\bar{\rho}_{(f)0}}{\phi_{(f)0}}h\hat{\bar{\rho}}_{(f)} + \hat{\lambda}, \tag{73}$$

$$ik\phi_{(s)0}\hat{u}_{(s)} + ik\phi_{(f)0}\hat{u}_{(f)} = \frac{\phi_{(s)0}}{\bar{\rho}_{(s)0}}\hat{\bar{\rho}}_{(s)} + \frac{\phi_{(f)0}}{\bar{\rho}_{(f)0}}\hat{\bar{\rho}}_{(f)}, \tag{74}$$

which are five homogeneous algebraic equations in the constants $\hat{u}_{(s)}$, $\hat{u}_{(f)}$, $\hat{\bar{\rho}}_{(s)}$, $\hat{\bar{\rho}}_{(f)}$ and $\hat{\lambda}$. Setting the determinant of the coefficients to zero gives the characteristic equation which can be solved for the phase velocity ω/Real (k) and the attenuation $-$Imaginary (k) as functions of frequency.

For the porous medium in the absence of fluid, the one-dimensional forms of (46) and (47) become

$$-\omega^2\rho_{(s)0}\hat{u}_{(s)} = -k^2(c+2d)\hat{u}_{(s)} - ikf\hat{\bar{\rho}}_{(s)}, \tag{75}$$

$$-\omega^2\bar{\rho}^2_{(s)0}I_{(s)0}\hat{\bar{\rho}}_{(s)} = ik\frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}}f\hat{u}_{(s)} - \frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}}g\hat{\bar{\rho}}_{(s)}. \tag{76}$$

The characteristic equation obtained from (75) and (76) can be solved in closed form to obtain

$$\omega/k = \left[\frac{(c+2d) + \left(\dfrac{f}{\rho_{(s)0}I_{(s)0}\omega^2 - g}\right)}{\rho_{(s)0}}\right]^{1/2}, \tag{77}$$

which is the phase velocity for compressional waves in the porous medium. Note that it is dispersive and exhibits a resonance at $\omega = (g/\rho_{(s)0}I_{(s)0})^{1/2}$. Also note that there is no attenuation.

It should be emphasized that the constants c, d, f and g retain the same numerical values whether or not fluid is present in the porous solid. Consequently for the same porous material the same numerical values would be used for these constants both in eqns (70)–(74) and in eqn (77).

The specific numerical results that are presented in this work are for Berea sandstone saturated with water. The properties of this material, including the Biot constants, are given by Yew and Jogi[20, 21]. The data are listed in Appendix 3 along with computed values of the expansional inertia coefficients $I_{(s)0}$ and $I_{(f)0}$ (see Appendix 1) and the constitutive coefficients $c+2d$, f and g.

The phase velocity and attenuation results computed from eqns (70) to (74) for water saturated Berea sandstone are shown in Figs. 1–6. The solid lines in these figures show the results for the present theory including both the virtual mass effect (see Appendix 2) and the expansional inertia effect. As in Biot's theory, two types of compressional waves exist in the material. They are termed the fast and slow waves.

As noted previously, the results of the present theory reduce to those predicted by Biot's theory when $I_{(s)0} = I_{(f)0} = 0$. These results are shown in the figures. The effects of assuming that $m = 0$ are also shown.

In Figs. 1–3 the results shown by dashed lines correspond to the Biot theory with and without the virtual mass effect. It is seen that the inclusion of this effect produces a pronounced change in the results. When the expansional inertia terms are also considered, even more pronounced changes are seen, particularly for frequencies above 1 MHz. Here the phase velocity drops sharply and the attenuation increases rapidly as the frequency approaches a resonant frequency of the unit cell.

Figures 4–6 illustrate how the virtual mass effect influences the present theory. Here again it is seen that large changes result from the inclusion of this term.

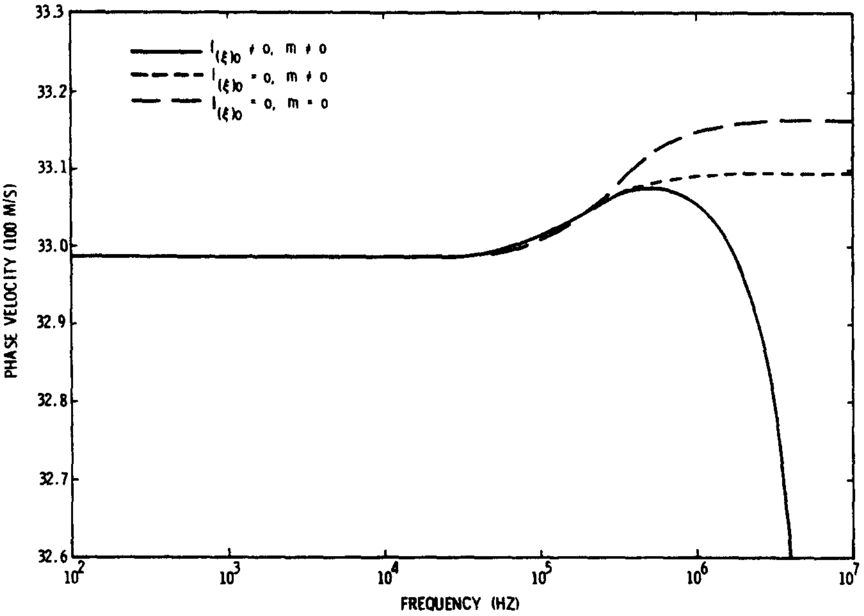

Fig. 1. Dispersion of the fast wave.

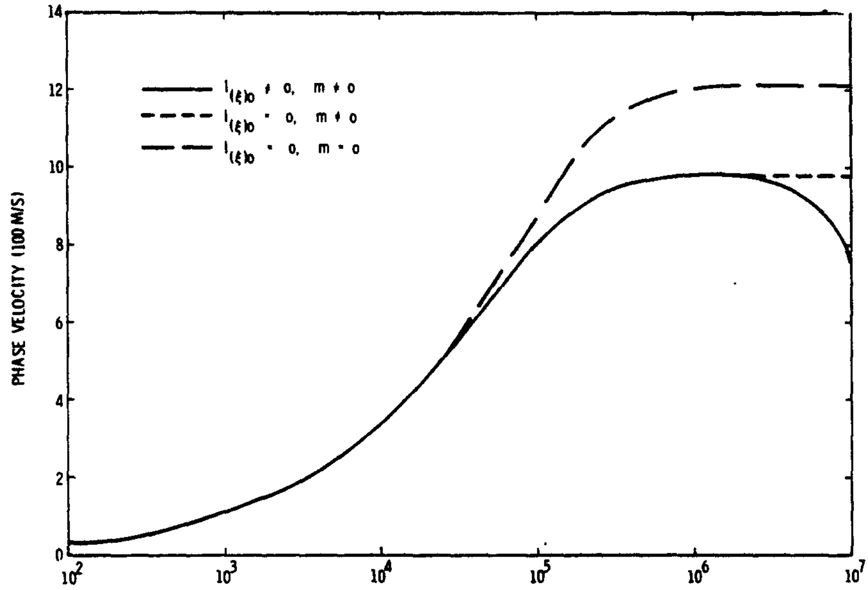

Fig. 2. Dispersion of the slow wave.

Results were also obtained for dry Berea sandstone using eqn (77). Again the parameters used were those listed in Appendix 3. However, in this case virtually no dispersion occurs; the phase velocity of the single wave is essentially constant with a value of 3207 m/s except in an extremely narrow frequency band near the resonance at 18.01 MHz.

When the latter value of phase velocity is compared to the low frequency value of the fast wave phase velocity for the saturated material, 3299 m/s, it is seen that the velocity drops by 3% when the water is removed. In Yew and Jogi[19], measured wet and dry velocities of 3970 m/s and 3818 m/s are listed. While there is substantial error between the experimental velocities and the computed velocities (which may be attributable to using Biot constants determined by static measurements[20] for the calculations), the experimentally determined shift in velocity between wet and dry conditions is 4%, which compares favorably with the calculated shift.

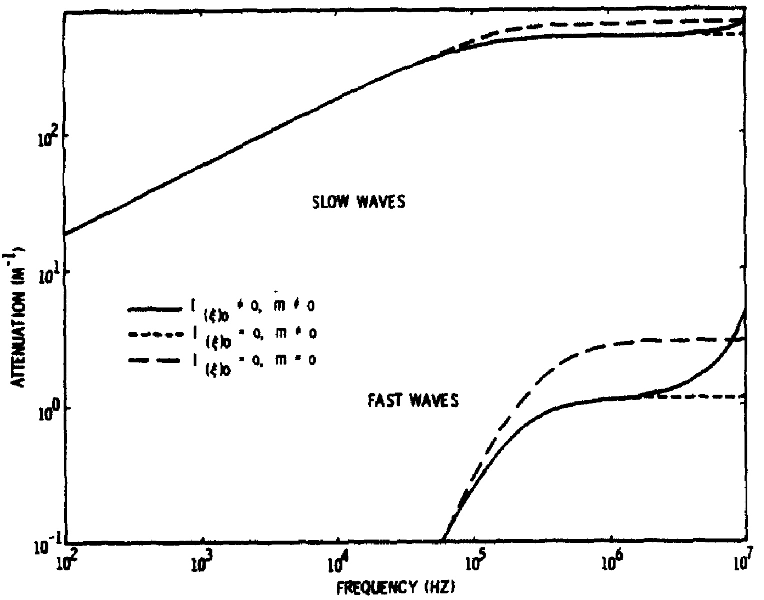

Fig. 3. Attenuation of the fast and slow waves.

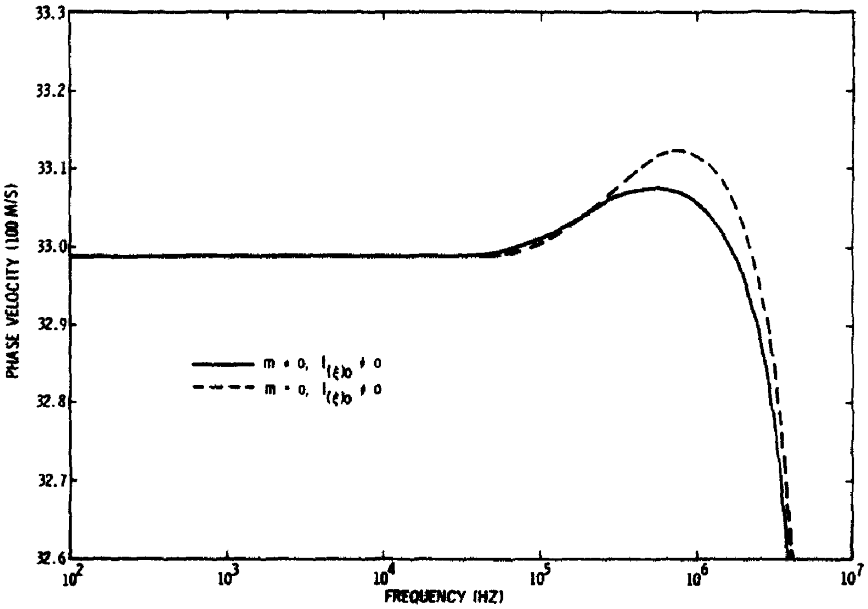

Fig. 4. Dispersion of the fast wave.

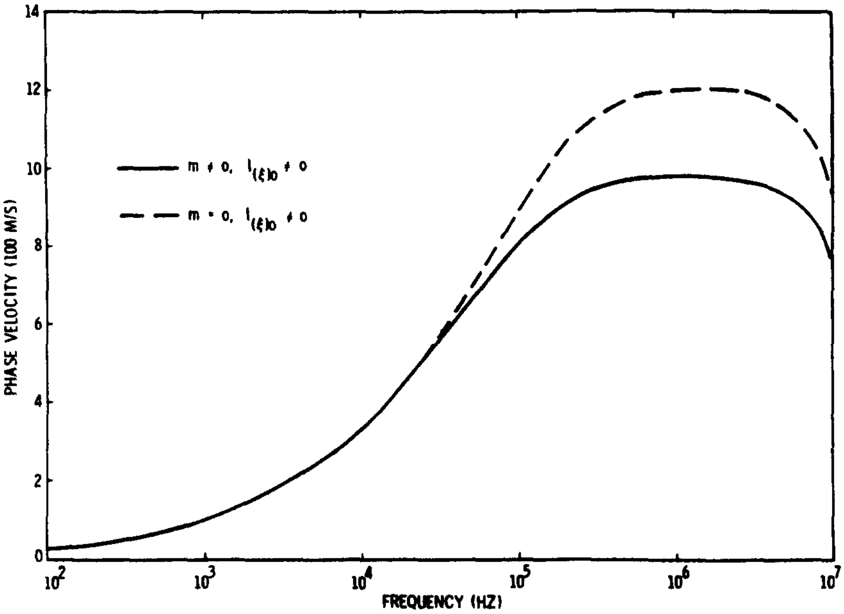

Fig. 5. Dispersion of the slow wave.

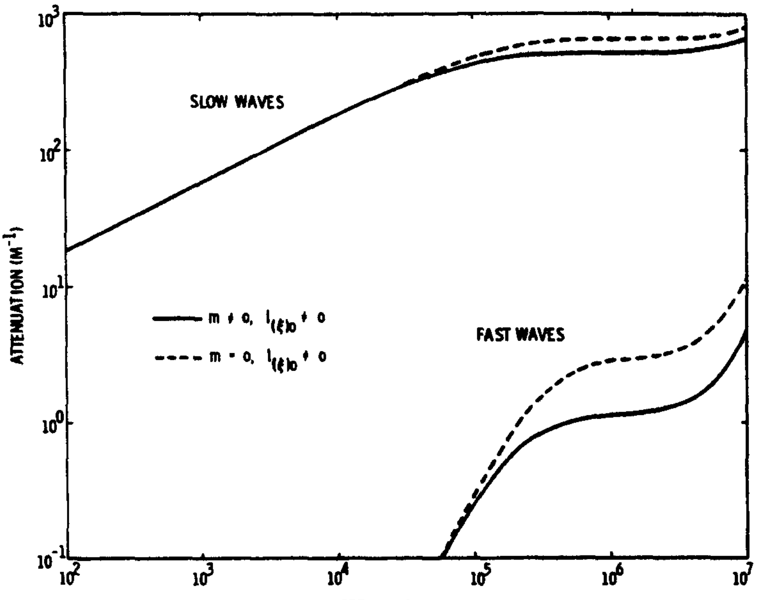

Fig. 6. Attenuation of the fast and slow waves.

In assessing the results that have been presented, it must be remembered that the coefficients $I_{(\xi)0}$ have been determined by modeling the porous structure as uniformly distributed cylinders of equal radius. In the real material, the pores will of course be randomly distributed and of random shape and size. Nevertheless, the results presented indicate that in the range of frequencies in which the fast wave in a fluid saturated porous medium exhibits dispersion, microinertial effects can substantially modify the dispersion and attenuation of the wave. This must be considered in experimental wave propagation studies.

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APPENDIX 1

Estimation of $I_{(\xi)}$

The coefficients $I_{(\xi)}$ in the local expansion kinetic energy term in eqn (16) are estimated by calculating the kinetic energy associated with a homogeneous expansion of a unit cell of the material. This corresponds to the breathing mode of the unit cell.

Let the material be modeled locally as consisting of parallel cylindrical pores of radius $r_{(f)}$ uniformly distributed in the solid material. The unit cell will be assumed to consist of a single pore and a cylinder of solid material of radius $r_{(s)}$ which encloses the pore (Fig. 7). The radius $r_{(s)}$ can be determined in terms of the pore radius and the volume fraction $\phi_{(f)}$:

$$(r_{(f)}/r_{(s)})^2 = \phi_{(f)}. \tag{1.1}$$

When the unit cell expands homogeneously in the radial direction, the radial velocity $\dot{r}$ of a material point in the fluid at

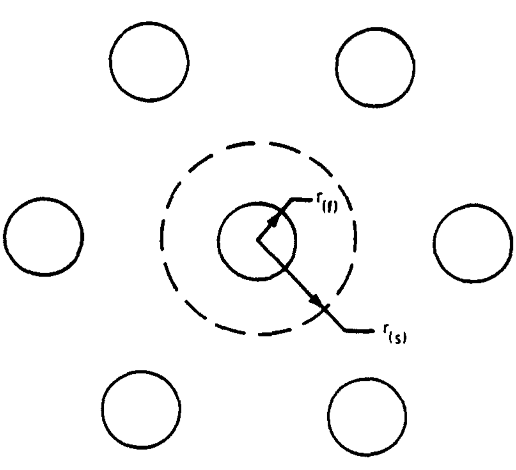

Fig. 7. The unit cell.

radial position r is

$$\dot{r} = \frac{r}{r_{(f)}}\dot{r}_{(f)}, \tag{1.2}$$

and the kinetic energy of the fluid in a portion of the pore of length L is

$$\int_0^{r_{(f)}} \frac{1}{2}\bar{\rho}_{(f)} 2\pi r\,\mathrm{d}r L\dot{r}^2 = \frac{1}{4}\pi\bar{\rho}_{(f)} L r_{(f)}^2 \dot{r}_{(f)}^2. \tag{1.3}$$

The number of cylinders per unit volume of the mixture is

$$\frac{\phi_{(f)}}{\pi r_{(f)}^2 L}. \tag{1.4}$$

Multiplying (1.3) and (1.4), the kinetic energy per unit volume is

$$\frac{1}{4}\phi_{(f)}\bar{\rho}_{(f)}\dot{r}_{(f)}^2. \tag{1.5}$$

For the homogeneous radial expansion,

$$\dot{r}_{(f)} = -\frac{r_{(f)}}{2\bar{\rho}_{(f)}}\dot{\bar{\rho}}_{(f)}, \tag{1.6}$$

so that (1.5) is

$$\frac{1}{16}\rho_{(f)}\frac{r_{(f)}^2}{\bar{\rho}_{(f)}^2}\dot{\bar{\rho}}_{(f)}^2. \tag{1.7}$$

Comparing (1.7) with the local expansion term in (16), it is seen that

$$I_{(f)} = \frac{r_{(f)}^2}{8\bar{\rho}_{(f)}^2}, \tag{1.8}$$

so that

$$I_{(f)0} = \frac{r_{(f)0}^2}{8\bar{\rho}_{(f)0}^2}. \tag{1.9}$$

A similar analysis for the cylinder of solid material gives the result

$$I_{(s)0} = \frac{(r_{(s)0}^2 + r_{(f)0}^2)}{8\bar{\rho}_{(s)0}^2}.. \tag{1.10}$$

APPENDIX 2

Variation of the relative velocity kinetic energy

The second term in eqn (16) is known as the virtual mass term. It was discussed by Biot[4] and is known to be important in particulate sedimentation and in the dynamics of bubbly liquids[22]. Its nature can be illustrated by a simple example.

Consider the porous solid-fluid mixture to be translating without deformation and with no relative velocity. The kinetic energy of the fluid is

$$\int_v \frac{1}{2}\rho_{(f)} v_{(f)k} v_{(f)k}\,\mathrm{d}v. \tag{2.1}$$

Next consider the fluid in motion with the solid stationary. The kinetic energy associated with the velocity of the fluid is, from (16),

$$\int_v \frac{1}{2}(\rho_{(f)} + m) v_{(f)k} v_{(f)k}\,\mathrm{d}v. \tag{2.2}$$

The energy (2.2) must be larger than (2.1) due to the sinuous motion of the fluid through the pores. Thus $m > 0$ and represents the energy associated with the motion of the fluid lateral to the direction of its mean motion when the relative velocity between the fluid and the porous solid is non-zero. For simple model pore geometries, it may be feasible to calculate m by means of this example.

Now consider the variation of the virtual mass term†

$$I = \int_{t_1}^{t_2}\int_v \frac{1}{2} m (v_{(f)k} - v_{(s)k})(v_{(f)k} - v_{(s)k})\,\mathrm{d}v\,\mathrm{d}t. \tag{2.3}$$

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†In a nonlinear theory, the coefficient m would be assumed to be a function of the state variables $\phi_{(f)}$, $\bar{\rho}_{(f)}$. Since this paper is concerned only with the linear theory, m is treated as a constant.

Using (14) and (12),

$$\delta I = \int_{t_1}^{t_2}\int_v m(v_{(f)k} - v_{(s)k})(\delta v_{(f)k}|_x - \delta v_{(s)k}|_x)\,\mathrm{d}v\,\mathrm{d}t \tag{2.4}$$

$$\begin{aligned} &= \int_{t_1}^{t_2}\int_v m(v_{(f)k} - v_{(s)k})\delta\dot{x}_{(f)k}\,\mathrm{d}v\,\mathrm{d}t - \int_{t_1}^{t_2}\int_v m(v_{(f)k} - v_{(s)k})\delta\dot{x}_{(s)k}\,\mathrm{d}v\,\mathrm{d}t \\ &\quad - \int_{t_1}^{t_2}\int_v m(v_{(f)k} - v_{(s)k})(v_{(f)k,j}\delta x_{(f)j} - v_{(s)k,j}\delta x_{(s)j})\,\mathrm{d}v\,\mathrm{d}t. \end{aligned} \tag{2.5}$$

Integrating the first two integrals in (2.5) by parts and using (4),

$$\begin{aligned} \delta I = &- \int_{t_1}^{t_2}\int_v m[(\dot{v}_{(f)k} - \dot{v}_{(s)k}|_{x_{(f)}}) + (v_{(f)k} - v_{(s)k})v_{(f)j,j}]\delta x_{(f)k}\,\mathrm{d}v\,\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int_v m[(\dot{v}_{(f)k}|_{x_{(s)}} - \dot{v}_{(s)k}) + (v_{(f)k} - v_{(s)k})v_{(s)j,j}]\delta x_{(s)k}\,\mathrm{d}v\,\mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_v m(v_{(f)k} - v_{(s)k})(v_{(f)k,j}\delta x_{(f)j} - v_{(s)k,j}\delta x_{(s)j})\,\mathrm{d}v\,\mathrm{d}t \end{aligned} \tag{2.6}$$

$$\begin{aligned} = &- \int_{t_1}^{t_2}\int_v m[a_{(f)k} - a_{(s)k} + (v_{(f)j,k} - v_{(s)k,j})(v_{(f)j} - v_{(s)j}) \\ &+ v_{(f)j,j}(v_{(f)k} - v_{(s)k})]\delta x_{(f)k}\,\mathrm{d}v\,\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int_v m[a_{(f)k} - a_{(s)k} + (v_{(s)j,k} - v_{(f)k,j})(v_{(f)j} - v_{(s)j}) \\ &+ v_{(s)j,j}(v_{(f)k} - v_{(s)k})]\delta x_{(s)k}\,\mathrm{d}v\,\mathrm{d}t. \end{aligned} \tag{2.7}$$

When linearized, (2.7) reduces to

$$\delta I = \int_{t_1}^{t_2}\int_v - m(a_{(f)k} - a_{(s)k})(\delta x_{(f)k} - \delta x_{(s)k})\,\mathrm{d}v\,\mathrm{d}t, \tag{2.8}$$

which is the expression used in (23). However, two comments must be made in passing.

First, mixture theories which contain a relative acceleration term have been criticized on the grounds that the relative acceleration does not satisfy the principle of material frame indifference[13]. However, it is easy to show that the terms in (2.7) are frame indifferent. That such combinations are frame indifferent was first shown by Adkins[23]. In the linear approximation, they reduce to the relative acceleration. Indeed, it has been pointed out by Bowen[13, Section 2.1] that the relative acceleration terms in mixture theories should be regarded as approximations of properly indifferent forms.

Furthermore, it has come to the attention of the authors that relative acceleration terms are being included in mixture theories that are applied to problems in which large velocity gradients and velocities occur. In such cases the full indifferent terms should be used, as the neglected terms are of the order of the convective accelerations and could substantially alter the solutions.

APPENDIX 3

Material constants

The following Biot constants for Berea sandstone as determined by static measurements are listed by Yew and Jogi[20, 21]:

$$\begin{aligned} P &= 22.879\ \mathrm{GPa}, \\ Q &= 1.0473\ \mathrm{GPa}, \\ R &= 0.33095\ \mathrm{GPa}, \\ \bar{\rho}_{(f)0} &= 1.0052\ \mathrm{Mg/m^3}, \\ \bar{\rho}_{(s)0} &= 2.6286\ \mathrm{Mg/m^3}, \\ m &= 0.094\ \mathrm{Mg/m^3}, \\ b &= 0.28128\ \mathrm{GPa-s/m^3}, \\ \phi_{(f)0} &= 0.1867. \end{aligned}$$

Also the fluid bulk modulus and the pore radius are

$$\begin{aligned} \lambda_{(f)} &= 2.2\ \mathrm{GPa}, \\ r_{(f)} &= 46.5\ \mu\mathrm{m}. \end{aligned}$$

From these values, using eqns (65)–(68) we compute

$$\begin{aligned} I_{(s)0} &= 2.4892 \times 10^{-16}\ \mathrm{m^8/kg^2}, \\ I_{(f)0} &= 2.6731 \times 10^{-16}\ \mathrm{m^8/kg^2}, \\ c + 2d &= 21.982\ \mathrm{GPa}, \\ f &= 35.932\ \mathrm{TPa}, \\ g &= 0.53406\ \mathrm{EPa}. \end{aligned}$$